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U(1)-Gauge Theory of Fermions in Spacetimes with Horizons

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A U(1)-gauge theory of fermions is obtained in spacetimes having horizons, including physically interesting black-hole spacetimes as well as unphysical spacetimes like NUT.

1. INTRODUCTION

Recently Dariescu *et al.* [1] obtained a U(1)-gauge theory for massive fermionic fields minimally coupled to a space–time described by the Kerr–Newman black hole. In this paper, we extend this U(1)-gauge theory to any space–time having horizons. This includes, among others, the physically interesting black-hole spacetimes as well as the NUT space–time, which is sometimes considered unphysical.

2. THE BACKGROUND SPACE–TIME

To extend the U(1) gauge theory of fermions to any space–time having horizons, we consider a general type of space–time given by

$$ds^2 = \frac{b^2 + q^2}{X}\, dp^2 + \frac{X}{p^2 + q^2}\,(d\tau + q^2\, d\sigma)^2 + \frac{p^2 + q^2}{Y}\, dq^2 - \frac{Y}{p^2 + q^2}\,(d\tau - p^2\, d\sigma)^2 \tag{1}$$

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where

$$X(p) = b - g^2 + 2np - \varepsilon p^2 - \frac{1}{3}\lambda p^4 \tag{2}$$

$$Y(q) = b + e^2 - 2nq + \varepsilon q^2 - \frac{1}{3}\lambda q^4 \tag{3}$$

with electric potential

$$A_\mu dx^\mu = \frac{eq}{p^2 + q^2}(d\tau - p^2\, d\sigma) \tag{4}$$

The space–time given by (1) was studied in detail by Plebanski [2]. Besides the cosmological constant λ, the space-time includes six real parameters: m and n are the mass and the NUT (or magnetic mass) parameters; b and ε are related to the angular momentum per unit mass and acceleration; e and g are the electric and magnetic charges. The surface $Y(q) = 0$ gives the horizons of the space–time. In the appropriate limits the space–time gives the combined NUT–Kerr–Newman–Kasuya–de Sitter space–time such that

$$\begin{aligned} ds^2 = {} & \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\exists^{-2}\Delta_\theta \sin^2\theta}{\Sigma}(a\ dt - \rho\ d\phi)^2 \\ & - \frac{\exists^{-2}\Delta_r}{\Sigma}(dt - A\ d\phi)^2 \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Sigma &= r^2 + (n + a\cos\theta)^2 \\ \Delta_\theta &= 1 + \frac{\lambda}{3} a^2 \cos^2\theta \\ \Delta_r &= (r^2 + a^2 + n^2)\left[1 - \frac{1}{3}\lambda(r^2 + 5n^2)\right] - 2(mr + n^2) + e^2 + g^2 \\ \exists &= 1 + \frac{1}{3}\lambda a^2 \\ \rho &= r^2 + a^2 + n^2 \\ A &= a\sin^2\theta - 2n\cos\theta \end{aligned} \tag{6}$$

We call the space–times given by (5) the hot NUT–Kerr–Newman–Kasuya space–time (H-NUT-KN-K), since the de Sitter space–time has been interpreted as being hot [3].

3. EXTENSION OF U(1)-GAUGE THEORY

In terms of the rigid frame $\{e_a\}$ whose dual $\{\omega^a\}$ are the null complex tetrads, the spacetimes given by (1) can be written as

$$ds^2 = 2(\omega^1\omega^2 - \omega^3\omega^4) = g_{ab}\omega^a\omega^b \tag{7}$$

where

$$\begin{aligned}
\omega^1 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}\,dp + i\sqrt{\frac{X}{p^2+q^2}}\,(d\tau + q^2\,d\sigma)\right]\\
\omega^2 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}\,dp - i\sqrt{\frac{X}{p^2+q^2}}\,(d\tau + q^2\,d\sigma)\right]\\
\omega^3 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\,(d\tau - p^2\,d\sigma) - \sqrt{\frac{p^2+q^2}{Y}}\,dq\right]\\
\omega^4 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\,(d\tau - p^2\,d\sigma) + \sqrt{\frac{p^2+q^2}{Y}}\,dq\right]
\end{aligned} \tag{8}$$

and

$$(g_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{9}$$

The U(1)-gauge invariant Lagrangian L for the massive fermionic fields ψ is given by [4]

$$L = \overline{\Psi}\gamma^\mu D_\mu \Psi + M\overline{\Psi}\Psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{10}$$

where

$$D_\mu = \delta_\mu \Psi + igA_\mu \Psi \tag{11}$$

and its h.c., where $D_\mu\Psi$ denotes the Levi-Civita covariant derivative. The last term in (10) corresponds to the Maxwell Lagrangian with the U(1)-gauge field tensor

$$F^{\mu\nu} = g^{\mu\alpha}\partial_\alpha A^\nu - g^{\nu\alpha}\partial_\alpha A^\mu - (g^{\mu\alpha}\partial_\alpha g^{\nu\beta} - g^{\nu\alpha}\partial_\alpha g^{\mu\beta})g_{\beta\sigma}A^\sigma \tag{12}$$

Then the Dirac-type equation gets the covariant expression

$$\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\Psi - \frac{1}{4}\Gamma_{\alpha\beta\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\Psi + M\Psi = 0 \tag{13}$$

and the Maxwell equations with sources have the standard form

$$\frac{1}{\sqrt{-g}}\partial_{\mu}[\sqrt{-g}F^{\nu\mu}] = J^{\nu} \tag{14}$$

We shall now put the U(1)-gauge theory of a massive fermionic field in the curved space–times given by (7). Using for the covarant derivative (11) the general expression

$$D_a\Psi = \Delta_a\Psi + igA_a\Psi \tag{15}$$

and its h.c., we write the Lagrangian (10) as

$$L = \overline{\Psi}\gamma^a D_a\Psi + M\overline{\Psi}\Psi + \frac{1}{4}F_{ab}F^{ab} \tag{16}$$

where the electromagnetic tensor F^{ab} can be expressed in terms of the Boyer coordinates

$$\{x^{\mu}\} = \{p, \sigma, q, \tau\} \tag{17}$$

$$F^{ab} = \omega^a_{\mu}\,\omega^b_{\nu}\,F^{\mu\nu} \tag{18}$$

Then the components of $F^{\mu\nu}$ are explicitly

$$\begin{aligned} F^{12} = {} & \frac{X}{p^2+q^2}A^2_{,p} - \frac{p^2+q^2}{q^4X-p^4Y}A^1_{,\sigma} - \frac{p^2+q^2}{q^2X+p^2Y}A^1_{,\tau} \\ & - \left[\frac{X(q^4X-p^4Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}\right. \\ & \left. + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^2 \\ & - \left[\frac{X(X-Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}\right]A^4 \end{aligned} \tag{19a}$$

$$F^{13} = \frac{X}{p^2+q^2}A^3_{,p} - \frac{Y}{p^2+q^2}A^1_{,q} - \frac{X}{Y}\left(\frac{Y}{p^2+q^2}\right)_{,p}A^3 + \frac{Y}{X}\left(\frac{X}{p^2+q^2}\right)_{,q}A^1 \tag{19b}$$

$$F^{14} = \frac{X}{p^2+q^2} A^4_{,p} - \frac{p^2+q^2}{q^2X+p^2Y} A^1_{,\sigma} - \frac{p^2+q^2}{X-Y} A^1_{,\tau}$$
$$- \left[\frac{X(q^4X-p^4Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X-p^2Y} \right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_{,p} \right] A^2$$
$$- \left[\frac{X(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X+p^2Y} \right)_{,p} \right] A^4 \tag{19c}$$

$$F^{23} = -\frac{Y}{p^2+q^2} A^2_{,p} + \frac{p^2+q^2}{q^4X-p^4Y} A^3_{,\sigma} + \frac{p^2+q^2}{q^2X+p^2Y} A^3_{,\tau}$$
$$+ \left[\frac{Y(q^4X-p^4Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^4X-p^4Y} \right)_{,q} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X+p^2Y} \right)_{,q} \right] A^2$$
$$+ \left[\frac{Y(q^2X-p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^4X-p^4Y} \right)_{,q} + \frac{X(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X+p^2Y} \right)_{,q} \right] A^4 \tag{19d}$$

$$F^{24} = \frac{p^2+q^2}{q^4X-p^4Y} A^4_{,\sigma} + \frac{p^2+q^2}{q^2X+p^2Y} A^4_{,\tau} - \frac{p^2+q^2}{q^2X+p^2Y} A^2_{,\sigma} - \frac{p^2+q^2}{X-Y} A^2_{,\tau} \tag{19e}$$

$$F^{34} = \frac{Y}{p^2+q^2} A^2_{,q} - \frac{p^2+q^2}{q^2X+p^2Y} A^3_{,\sigma} - \frac{p^2+q^2}{X-Y} A^3_{,\tau}$$
$$- \left[\frac{Y(q^4X-p^4Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X-p^2Y} \right) + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_{,q} \right] A^2$$
$$- \left[\frac{Y(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_{,q} + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2X+p^2Y} \right)_{,q} \right] A^4 \tag{19f}$$

Consequently Maxwell's equations (14) can be written as

$$e_a F^{ab} = J^b \tag{20}$$

Now we write the Dirac equation for massive field Ψ coupled to the space–time concerned. Working out the U(1)-gauge invariant Lagrangian (16), we can write the Dirac-type equation in the general form as

$$\gamma^a(e_a + igA_a)\Psi - \frac{1}{4}\Gamma_{bca}\gamma^a\gamma^b\gamma^c + M\Psi = 0 \tag{21}$$

Equation (21), in the case of the spacetimes (7), will be reduced to the form

$$\gamma^a(e_a + igA_a)\Psi + M\Psi - \frac{1}{4}\frac{1}{\sqrt{2(p^2+q^2)^3}}$$
$$\times \left\{ \left[-\frac{(p^2+q^2)(\partial X/\partial p) - 2pX}{\sqrt{X}}(\gamma^1+\gamma^2) \right.\right.$$
$$\left. + \frac{(p^2+q^2)(\partial Y/\partial q) - 2qY}{\sqrt{Y}}(\gamma^3-\gamma^4) \right]$$
$$\left. - 8i[p\sqrt{Y}\,\gamma^1\gamma^2(\gamma^3+\gamma^4) - q\sqrt{X}\,(\gamma^1-\gamma^2)\gamma^3\gamma^4] \right\}\Psi = 0 \qquad (22)$$

4. DISCUSSION

For $n = 0$, equation (22) stands for all the dyon black-hole solutions.

In special cases equation (22) stands for all the dyon black-hole solutions generalized with the NUT parameter. The result is interesting in that the spacetimes endowed with the NUT parameter should never be physically interpreted [5]. The result will be more interesting if we specialize the result for the NUT space–time, which is sometimes considered as unphysical [6]. So it is interesting to note that we set the U(1)-gauge theory of fermions not only in the physically interesting black-hole solutions, but also in space–times which have no physical interpretation. This result obtained in such diverse situations is possible only when the space–times have horizons.

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